

## 4. Linear Systems

### 4A. Review of Matrices

**4A-1.** Verify that  $\begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \end{pmatrix}$ .

**4A-2.** If  $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ , show that  $AB \neq BA$ .

**4A-3.** Calculate  $A^{-1}$  if  $A = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$ , and check your answer by showing that  $AA^{-1} = I$  and  $A^{-1}A = I$ .

**4A-4.** Verify the formula given in Notes LS.1 for the inverse of a  $2 \times 2$  matrix.

**4A-5.** Let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ . Find  $A^3 (= A \cdot A \cdot A)$ .

**4A-6.** For what value of  $c$  will the vectors  $\mathbf{x}_1 = (1, 2, c)$ ,  $\mathbf{x}_2 = (-1, 0, 1)$ , and  $\mathbf{x}_3 = (2, 3, 0)$  be linearly dependent? For this value, find by trial and error (or otherwise) a linear relation connecting them, i.e., one of the form  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$

### 4B. General Systems; Elimination; Using Matrices

**4B-1.** Write the following equations as equivalent first-order systems:

a)  $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + tx^2 = 0$                       b)  $y'' - x^2y' + (1 - x^2)y = \sin x$

**4B-2.** Write the IVP

$$y^{(3)} + p(t)y'' + q(t)y' + r(t)y = 0, \quad y(0) = y_0, \quad y'(0) = y'_0, \quad y''(0) = y''_0$$

as an equivalent IVP for a system of three first-order linear ODE's. Write this system both as three separate equations, and in matrix form.

**4B-3.** Write out  $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$ ,  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$  as a system of two first-order equations.

a) Eliminate  $y$  so as to obtain a single second-order equation for  $x$ .

b) Take the second-order equation and write it as an equivalent first-order system. This isn't the system you started with, but show a change of variables converts one system into the other.

**4B-4.** For the system  $x' = 4x - y$ ,  $y' = 2x + y$ ,

a) using matrix notation, verify that  $x = e^{3t}$ ,  $y = e^{3t}$  and  $x = e^{2t}$ ,  $y = 2e^{2t}$  are solutions;

b) verify that they form a fundamental set of solutions — i.e., that they are linearly independent;

c) write the general solution to the system in terms of two arbitrary constants  $c_1$  and  $c_2$ ; write it both in vector form, and in the form  $x = \dots$ ,  $y = \dots$ .

**4B-5.** For the system  $\mathbf{x}' = A\mathbf{x}$ , where  $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ ,

a) show that  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$  and  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$  form a fundamental set of solutions (i.e., they are linearly independent and solutions);

b) solve the IVP:  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ .

**4B-6.** Solve the system  $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$  in two ways:

a) Solve the second equation, substitute for  $y$  into the first equation, and solve it.

b) Eliminate  $y$  by solving the first equation for  $y$ , then substitute into the second equation, getting a second order equation for  $x$ . Solve it, and then find  $y$  from the first equation. Do your two methods give the same answer?

**4B-7.** Suppose a radioactive substance  $R$  decays into a second one  $S$  which then also decays. Let  $x$  and  $y$  represent the amounts of  $R$  and  $S$  present at time  $t$ , respectively.

a) Show that the physical system is modeled by a system of equations

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} -a & 0 \\ a & -b \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad a, b \text{ constants.}$$

b) Solve this system by either method of elimination described in 4B-6.

c) Find the amounts present at time  $t$  if initially only  $R$  is present, in the amount  $x_0$ .

**Remark.** The method of elimination which was suggested in some of the preceding problems (4B-3,6,7; book section 5.2) is always available. Other than in these exercises, we will not discuss it much, as it does not give insights into systems the way the methods will describe later do.

**Warning.** Elimination sometimes produces extraneous solutions — extra “solutions” that do not actually solve the original system. Expect this to happen when you have to differentiate both equations to do the elimination. (Note that you also get extraneous solutions when doing elimination in ordinary algebra, too.) If you get more independent solutions than the order of the system, they must be tested to see if they actually solve the original system. (The order of a system of ODE's is the sum of the orders of each of the ODE's in it.)

#### 4C. Eigenvalues and Eigenvectors

**4C-1.** Solve  $\mathbf{x}' = A\mathbf{x}$ , if  $A$  is: a)  $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$  b)  $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$  c)  $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ .

( First find the eigenvalues and associated eigenvectors, and from these construct the normal modes and then the general solution.)

**4C-2.** Prove that  $m = 0$  is an eigenvalue of the  $n \times n$  constant matrix  $A$  if and only if  $A$  is a singular matrix ( $\det A = 0$ ). (You can use the characteristic equation, or you can use the definition of eigenvalue.)

**4C-3.** Suppose a  $3 \times 3$  matrix is upper triangular. (This means it has the form below, where  $*$  indicates an arbitrary numerical entry.)

$$A = \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix}$$

Find its eigenvalues. What would be the generalization to an  $n \times n$  matrix?

**4C-4.** Show that if  $\vec{\alpha}$  is an eigenvector of the matrix  $A$ , associated with the eigenvalue  $m$ , then  $\vec{\alpha}$  is also an eigenvector of the matrix  $A^2$ , associated this time with the eigenvalue  $m^2$ . (Hint: use the eigenvector equation in 4F-3.)

**4C-5.** Solve the radioactive decay problem (4B-7) using eigenvectors and eigenvalues.

**4C-6.** Farmer Smith has a rabbit colony in his pasture, and so does Farmer Jones. Each year a certain fraction  $a$  of Smith's rabbits move to Jones' pasture because the grass is greener there, and a fraction  $b$  of Jones' rabbits move to Smith's pasture (for the same reason). Assume (foolishly, but conveniently) that the growth rate of rabbits is 1 rabbit (per rabbit/per year).

a) Write a system of ODE's for determining how  $S$  and  $J$ , the respective rabbit populations, vary with time  $t$  (years).

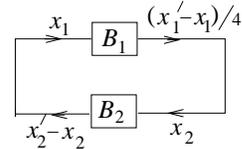
b) Assume  $a = b = \frac{1}{2}$ . If initially Smith has 20 rabbits and Jones 10 rabbits, how do the two populations subsequently vary with time?

c) Show that  $S$  and  $J$  never oscillate, regardless of  $a$ ,  $b$  and the initial conditions.

**4C-7.** The figure shows a simple feedback loop.

Black box  $B_1$  inputs  $x_1(t)$  and outputs  $\frac{1}{4}(x_1' - x_1)$ .

Black box  $B_2$  inputs  $x_2(t)$  and outputs  $x_2' - x_2$ .



If they are hooked up in a loop as shown, and initially  $x_1 = 1, x_2 = 0$ , how do  $x_1$  and  $x_2$  subsequently vary with time  $t$ ? (If it helps, you can think of  $x_1$  and  $x_2$  as currents, for instance, or as the monetary values of trading between two countries, or as the number of times/minute Punch hits Judy and vice-versa.)

#### 4D. Complex and Repeated Eigenvalues

**4D-1.** Solve the system  $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ .

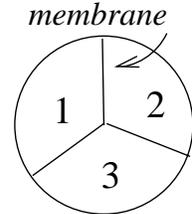
**4D-2.** Solve the system  $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}$ .

**4D-3.** Solve the system  $\mathbf{x}' = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$ .

**4D-4.** Three identical cells are pictured, each containing salt solution, and separated by identical semi-permeable membranes. Let  $A_i$  represent the amount of salt in cell  $i$  ( $i = 1, 2, 3$ ), and let

$$x_i = A_i - A_0$$

be the difference between this amount and some standard reference amount  $A_0$ . Assume the rate at which salt diffuses across the membranes is proportional to the difference in concentrations, i.e. to the difference in the two values of  $A_i$  on either side, since we are supposing the cells identical. Take the constant of proportionality to be 1.



a) Derive the system  $\mathbf{x}' = A\mathbf{x}$ , where  $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ .

b) Find three normal modes, and interpret each of them physically. (To what initial conditions does each correspond — is it reasonable as a solution, in view of the physical set-up?)

#### 4E. Decoupling

**4E-1.** A system is given by  $x' = 4x + 2y$ ,  $y' = 3x - y$ . Give a new set of variables,  $u$  and  $v$ , linearly related to  $x$  and  $y$ , which decouples the system. Then verify by direct substitution that the system becomes decoupled when written in terms of  $u$  and  $v$ .

**4E-2.** Answer the same questions as in the previous problem for the system in 4D-4. (Use the solution given in the Notes to get the normal modes. In the last part of the problem, do the substitution by using matrices.)

#### 4F. Theory of Linear Systems

**4F-1.** Take the second-order equation  $x'' + p(t)x' + q(t)x = 0$ .

a) Change it to a first-order system  $\mathbf{x}' = A\mathbf{x}$  in the usual way.

b) Show that the Wronskian of two solutions  $x_1$  and  $x_2$  of the original equation is the same as the Wronskian of the two corresponding solutions  $\mathbf{x}_1$  and  $\mathbf{x}_2$  of the system.

**4F-2.** Let  $\mathbf{x}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$  be two vector functions.

a) Prove by using the definition that  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are linearly independent.

b) Calculate the Wronskian  $W(\mathbf{x}_1, \mathbf{x}_2)$ .

c) How do you reconcile (a) and (b) with Theorem 5C in Notes LS.5?

d) Find a linear system  $\mathbf{x}' = A\mathbf{x}$  having  $\mathbf{x}_1$  and  $\mathbf{x}_2$  as solutions, and confirm your answer to (c). (To do this, treat the entries of  $A$  as unknowns, and find a system of equations whose solutions will give you the entries.  $A$  will be a matrix function of  $t$ , i.e., its entries will be functions of  $t$ .)

**4F-3.** Suppose the  $2 \times 2$  constant matrix  $A$  has two distinct eigenvalues  $m_1$  and  $m_2$ , with associated eigenvectors respectively  $\vec{\alpha}_1$  and  $\vec{\alpha}_2$ . Prove that the corresponding vector functions

$$\mathbf{x}_1 = \vec{\alpha}_1 e^{m_1 t}, \quad \mathbf{x}_2 = \vec{\alpha}_2 e^{m_2 t}$$

are linearly independent, as follows:

a) using the determinantal criterion, show they are linearly independent if and only if  $\vec{\alpha}_1$  and  $\vec{\alpha}_2$  are linearly independent;

b) then show that  $c_1 \vec{\alpha}_1 + c_2 \vec{\alpha}_2 = \mathbf{0} \Rightarrow c_1 = 0, c_2 = 0$ . (Use the eigenvector equation  $(A - m_i I)\vec{\alpha}_i = \mathbf{0}$  in the form:  $A\vec{\alpha}_i = m_i \vec{\alpha}_i$ .)

**4F-4.** Suppose  $\mathbf{x}' = A\mathbf{x}$ , where  $A$  is a nonsingular constant matrix. Show that if  $\mathbf{x}(t)$  is a solution whose velocity vector  $\mathbf{x}'(t)$  is  $\mathbf{0}$  at time  $t_0$ , then  $\mathbf{x}(t)$  is identically zero for all  $t$ . What is the minimum hypothesis on  $A$  that is needed for this result to be true? Can  $A$  be a function of  $t$ , for example?

#### 4G. Fundamental Matrices

**4G-1.** Two independent solutions to  $\mathbf{x}' = A\mathbf{x}$  are  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$  and  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$ .

a) Find the solutions satisfying  $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

b) Using part (a), find in a simple way the solution satisfying  $\mathbf{x}(0) = \begin{pmatrix} a \\ b \end{pmatrix}$ .

**4G-2.** For the system  $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$ ,

a) find a fundamental matrix, using the normal modes, and use it to find the solution satisfying  $x(0) = 2, y(0) = -1$ ;

b) find the fundamental matrix normalized at  $t = 0$ , and solve the same IVP as in part (a) using it.

**4G-3.\*** Same as 4G-2, using the matrix  $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$  instead.

#### 4H. Exponential Matrices

**4H-1.** Calculate  $e^{At}$  if  $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ . Verify directly that  $\mathbf{x} = e^{At}\mathbf{x}_0$  is the solution to  $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \mathbf{x}_0$ .

**4H-2.** Calculate  $e^{At}$  if  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ; then answer same question as in 4H-1.

**4H-3.** Calculate  $e^{At}$  directly from the infinite series, if  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ; then answer same question as in 4H-1.

**4H-4.** What goes wrong with the argument justifying the  $e^{At}$  solution of  $\mathbf{x}' = A\mathbf{x}$  if  $A$  is not a constant matrix, but has entries which depend on  $t$ ?

**4H-5.** Prove that a)  $e^{kIt} = Ie^{kt}$ .      b)  $Ae^{At} = e^{At}A$ .

(Here  $k$  is a constant,  $I$  is the identity matrix,  $A$  any square constant matrix.)

**4H-6.** Calculate the exponential matrix in 4H-3, this time using the third method in the Notes (writing  $A = B + C$ ).

**4H-7.** Let  $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ . Calculate  $e^{At}$  three ways:

- a) directly, from its definition as an infinite series;
- b) by expressing  $A$  as a sum of simpler matrices, as in Notes LS.6, Example 6.3C;
- c) by solving the system by elimination so as to obtain a fundamental matrix, then normalizing it.

#### 4I. Inhomogeneous Systems

**4I-1.** Solve  $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \end{pmatrix} t$ , by variation of parameters.

**4I-2.** a) Solve  $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$  by variation of parameters.

b) Also do it by undetermined coefficients, by writing the forcing term and trial solution respectively in the form:

$$\begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t; \quad \mathbf{x}_p = \vec{c}e^{-2t} + \vec{d}e^t.$$

**4I-3.\*** Solve  $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$  by undetermined coefficients.

**4I-4.** Solve  $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$  by undetermined coefficients.

**4I-5.** Suppose  $\mathbf{x}' = A\mathbf{x} + \mathbf{x}_0$  is a first-order order system, where  $A$  is a nonsingular  $n \times n$  constant matrix, and  $\mathbf{x}_0$  is a constant  $n$ -vector. Find a particular solution  $\mathbf{x}_p$ .

## Section 5. Graphing Systems

### 5A. The Phase Plane

**5A-1.** Find the critical points of each of the following non-linear autonomous systems.

$$\begin{array}{ll} \text{a)} & \begin{cases} x' = x^2 - y^2 \\ y' = x - xy \end{cases} & \text{b)} & \begin{cases} x' = 1 - x + y \\ y' = y + 2x^2 \end{cases} \end{array}$$

**5A-2.** Write each of the following equations as an equivalent first-order system, and find the critical points.

$$\text{a)} \quad x'' + a(x^2 - 1)x' + x = 0 \qquad \text{b)} \quad x'' - x' + 1 - x^2 = 0$$

**5A-3.** In general, what can you say about the relation between the trajectories and the critical points of the system on the left below, and those of the two systems on the right?

$$\begin{array}{lll} \begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} & \text{a)} & \begin{cases} x' = -f(x, y) \\ y' = -g(x, y) \end{cases} & \text{b)} & \begin{cases} x' = g(x, y) \\ y' = -f(x, y) \end{cases} \end{array}$$

**5A-4.** Consider the autonomous system

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}; \quad \text{solution: } \mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

a) Show that if  $\mathbf{x}_1(t)$  is a solution, then  $\mathbf{x}_2(t) = \mathbf{x}_1(t - t_0)$  is also a solution. What is the geometric relation between the two solutions?

b) The existence and uniqueness theorem for the system says that if  $f$  and  $g$  are continuously differentiable everywhere, there is one and only one solution  $\mathbf{x}(t)$  satisfying a given initial condition  $\mathbf{x}(t_0) = \mathbf{x}_0$ .

Using this and part (a), show that two trajectories cannot intersect anywhere.

(Note that if two trajectories intersect at a point  $(a, b)$ , the corresponding solutions  $\mathbf{x}(t)$  which trace them out may be at  $(a, b)$  at different times. Part (a) shows how to get around this difficulty.)

### 5B. Sketching Linear Systems

**5B-1.** Follow the Notes (GS.2) for graphing the trajectories of the system  $\begin{cases} x' = -x \\ y' = -2y \end{cases}$ .

a) Eliminate  $t$  to get one ODE  $\frac{dy}{dx} = F(x, y)$ . Solve it and sketch the solution curves.

b) Solve the original system (by inspection, or using eigenvalues and eigenvectors), obtaining the equations of the trajectories in parametric form:  $x = x(t), y = y(t)$ . Using these, put the direction of motion on your solution curves for part (a). What new trajectories are there which were not included in the curves found in part (a)?

c) How many trajectories are needed to cover a typical solution curve found in part (a)? Indicate them on your sketch.

d) If the system were  $x' = x$ ,  $y' = 2y$  instead, how would your picture be modified? (Consider both parts (a) and (b).)

**5B-2.** Answer the same questions as in 5B-1 for the system  $x' = y$ ,  $y' = x$ . (For part (d), use  $-y$  and  $-x$  as the two functions on the right.)

**5B-3.** Answer the same question as in 5B-1a,b for the system  $x' = y$ ,  $y' = -2x$ .

For part (b), put in the direction of motion on the curves by making use of the vector field corresponding to the system.

**5B-4.** For each of the following linear systems, carry out the graphing program in Notes GS.4; that is,

(i) find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;

(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing  $t$ ; then draw in some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

a) $\begin{aligned} x' &= 2x - 3y \\ y' &= x - 2y \end{aligned}$	b) $\begin{aligned} x' &= 2x \\ y' &= 3x + y \end{aligned}$	c) $\begin{aligned} x' &= -2x - 2y \\ y' &= -x - 3y \end{aligned}$
d) $\begin{aligned} x' &= x - 2y \\ y' &= x + y \end{aligned}$	e) $\begin{aligned} x' &= x + y \\ y' &= -2x - y \end{aligned}$	

**5B-5.** For the damped spring-mass system modeled by the ODE

$$mx'' + cx' + kx = 0, \quad m, c, k > 0,$$

a) write it as an equivalent first-order linear system;

b) tell what the geometric type of the critical point at  $(0, 0)$  is, and determine its stability, in each of the following cases. Do it by the methods of Sections GS.3 and GS.4, and check the result by physical intuition.

(i)  $c = 0$     (ii)  $c \approx 0$ ;  $m, k \gg 1$ .    (iii) Can the geometric type be a saddle? Explain.