به نام خدا

پاسخ سوالات امتحان میان ترم ریاضی مهندسی - همه‌ی گروه‌ها

آذر ماه ۱۳۹۵

۱- الف)

$$z^3+(z+2)^3=0$$

$$u=z$$

$$v=z+2$$

$$u^3+v^3=0 \longrightarrow (u+v)(u^2+v^2-uv)=0 \longrightarrow$$

$$(2z+2)(z^2+z^2+4z+4-z^2-2z)=0 \longrightarrow$$

$$(2z+2)(z^2+2z+4)=0 \longrightarrow$$

$$z=-1 \qquad z=-1\pm\sqrt{1-4}=-1\pm\sqrt{3}\,i$$

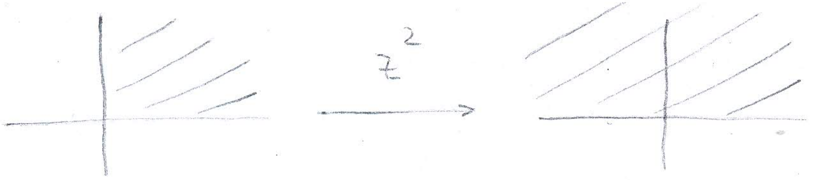
ب) در صورتی که بدانیم تابع $f(z)=u+iv$ مشتق پذیر است داریم

$$f'(z)=u_x+iv_x=u_x-iu_y \qquad u(x,y)=x^2-y^2+\frac{x}{x^2+y^2}$$

$$=2x+\underbrace{\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}}-i\underbrace{\left[-2y+\frac{x\times 2y\times(-1)}{(x^2+y^2)^2}\right]}$$

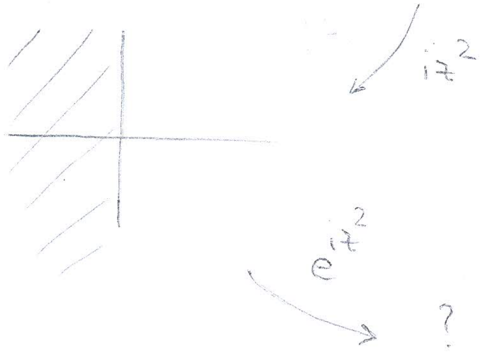
$$=2x+\frac{y^2-x^2}{(x^2+y^2)^2}+\left[2y+\frac{2xy}{(x^2+y^2)^2}\right]i$$

۲ - الف)




$0 < r < \infty$

$0 < \theta < \pi$

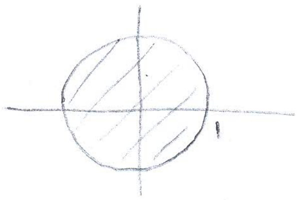



در واقع تصویر خطوط قائم $x=a$ برای $a<0$ تحت نگاشت $e^z$ مد نظر است


$$e^z = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

$$x=a \longrightarrow u = e^a \cos y,\quad v = e^a \sin y \longrightarrow u^2+v^2 = e^{2a}$$

دایره به مرکز مبدا و شعاع $e^a$ برای $a<0$




$a<0 \quad e^a < 1$

$\downarrow$

تصویر می‌شود داخل دایره‌ی واحد

همدیس بودن: نگاشت فوق در ناحیه‌ی داده شده همدیس است.

$f'(z) = 2iz e^{iz^2} = 0 \longrightarrow z=0$ جزو ناحیه نیست $\longrightarrow$

یک به یک بودن: واضح است که برای هر دو نقطه‌ی $z_1 = a+bi$ ، $z_2 = a+(b+2\pi)i$ داریم

$e^{z_1} = e^{z_2}$ بنابراین نگاشت $e^z$ یک به یک نیست

اما در مورد این نگاشت شرط یک به یک بودن را بررسی می‌کنیم:

$$e^{iz_1^2} = e^{iz_2^2} \longrightarrow e^{i(x_1^2-y_1^2+2x_1y_1 i)} = e^{i(x_2^2-y_2^2+2x_2y_2 i)}$$

$$\downarrow$$

$$e^{-2x_1y_1} = e^{-2x_2y_2} \longrightarrow x_1y_1 = x_2y_2$$

$$e^{i(x_1^2-y_1^2)} = e^{i(x_2^2-y_2^2)} \longrightarrow x_1^2-y_1^2 = x_2^2-y_2^2+2k\pi$$

$x_1$ ، $y_1$ ، $x_2$ ، $y_2$ ی وجود دارند که $x_1+y_1 i \neq x_2+y_2 i$ ولی $e^{iz_1^2} = e^{iz_2^2}$

پس تابع فوق یک به یک نیست

ب)

| | | | |
|---|---|---|---|
| $z_1$ | $0$ | $\longrightarrow$ | $i$ |
| $z_2$ | $1$ | $\longrightarrow$ | $\infty$ |
| $z_3$ | $i$ | $\longrightarrow$ | $1$ |

$$\frac{(z-0)(1-i)}{(z-i)(1-0)} = \frac{(w-i)(\cancel{\infty}-1)}{(w-1)(\cancel{\infty}-i)}$$

$$\lim = 1$$

$$\frac{(1-i)z}{z-i} = \frac{w-i}{w-1} \longrightarrow (1-i)z(w-1) = (z-i)(w-i)$$

$$w\left[(1-i)z-(z-i)\right] = (1-i)z - i(z-i)$$

$$w = \frac{(1-2i)z-1}{-iz+1}$$

$f(z) = \frac{1}{2}\left(\frac{1}{z-4} - \frac{1}{z-2}\right) \quad 2 < |z| < 4$ (الف ۳-

$f_1(z) = \frac{1}{z-4} = \frac{1}{4t-4} = -\frac{1}{4}\frac{1}{1-t} = -\frac{1}{4}\sum_{n=0}^{\infty} t^n = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$

$|z|<4 \quad t = \frac{z}{4} \quad |t|<1$

$f_2(z) = \frac{1}{z-2} = \frac{1}{z(1-\frac{2}{z})} = \frac{t/2}{1-t} = \frac{t}{2}\sum_{n=0}^{\infty} t^n = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

$|z|>2 \quad t = \frac{2}{z} \quad |t|<1$

$\longrightarrow f(z) = \frac{1}{2}(f_1(z) - f_2(z)) = -\frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$

$g(z) = \frac{1}{(z-2)^2} = -\left(\frac{1}{z-2}\right)' \longrightarrow$ (ب

$\frac{1}{z-2} = \frac{1}{2t-2} = -\frac{1}{2}\frac{1}{1-t} = -\frac{1}{2}\sum_{n=0}^{\infty} t^n = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

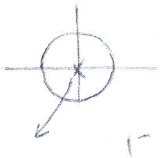
$|z|<2 \quad t = \frac{z}{2} \quad |t|<1$

$g(z) = -\left(\frac{1}{z-2}\right)' = -\left(-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right)' = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^{n-1} \times \frac{1}{2}$

$= \frac{1}{4}\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^{n-1}$

$$I_1 = \oint_{|z|=1/2} \frac{\sin 1/z}{1+z}\, dz$$

۴ - الف



$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \frac{1}{7!z^7} + \cdots$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \cdots$$

سری لوران $\Longrightarrow$ قطب اساسی

$$\sin 1/z \times \frac{1}{1+z} = \frac{1}{z}\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right) + \cdots$$

$$\underset{z=0}{\text{res}} \frac{\sin 1/z}{1+z} = 1 - \frac{1}{3!} + \frac{1}{5!} \cdots = \sin 1 \longrightarrow \oint_{\|z\|=1/2} \frac{\sin 1/z}{1+z} = 2\pi i \sin 1$$

ب)

$$I_2 = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} \qquad |a|<1$$

$$z = e^{i\theta} \longrightarrow \cos\theta = \frac{z + 1/z}{2}$$

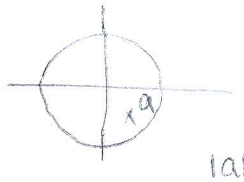
$$dz = ie^{i\theta}d\theta \longrightarrow d\theta = \frac{dz}{iz}$$

$\|z\| = 1 \rightarrow$

$$I_2 = \oint_{\|z\|=1} \frac{dz/iz}{1 - a(z + \frac{1}{z}) + a^2} = \oint_{\|z\|=1} \frac{dz}{i(z - az^2 - a + a^2 z)}$$

$$= \oint_{\|z\|=1} \frac{dz}{-ai(z^2 - (a + \frac{1}{a})z + 1)}$$

$$= \oint_{\|z\|=1} \frac{dz}{-ai(z-a)(z-1/a)}$$

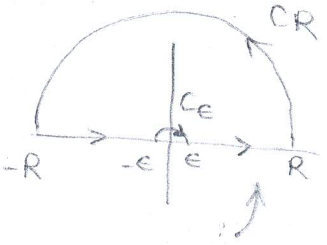


$|a|<1$

$$= 2\pi i \underset{z=a}{\text{Res}}$$

$$= 2\pi i \ast \frac{1}{-ai(z - \frac{1}{a})}\Big|_{z=a} = 2\pi\cancel{i} \frac{1}{-a\cancel{i}(a - \frac{1}{a})} = \frac{2\pi}{1-a^2}$$

$$I_3 = \int_{-\infty}^{\infty} \frac{dx}{x(x^2-2x+2)} \qquad f(z) = \frac{1}{z(z^2-2z+2)}$$

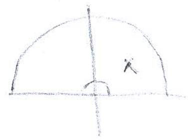


$$\oint_C f(z)\,dz = \int_{C_R} + \int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^{R}$$

$R \to \infty \quad \epsilon \to 0$

$$\oint_C = \int_{C_\infty} + \int_{-\infty}^{\infty} + \int_{C_\epsilon} \leftarrow$$

$$\oint_C f(z)\,dz = ?$$



$$z^2 - 2z + 2 = 0 \to z = 1 \pm \sqrt{1-2} = 1 \pm i$$

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1+i} f(z) = 2\pi i \left.\frac{1}{z(z-1+i)}\right|_{1+i}$$

$$= 2\pi i \frac{1}{(1+i)(2i)} = \frac{\pi}{1+i} = \frac{\pi(1-i)}{2} \leftarrow$$

$$\int_{C_\epsilon} f(z)\,dz = -\pi i \operatorname*{Res}_{z=0} f(z) = -\pi i \left.\frac{1}{z^2-2z+2}\right|_{z=0} = -\frac{\pi i}{2} \leftarrow$$

$$\int_{C_\infty} f(z)\,dz \overset{?}{=} 0$$

$$\left|\int_{C_R}\right| \leq ML \qquad L(C_R) = \pi R \leftarrow$$

$$M = \max_{C_R} |f| = \max_{|z|=R} \left|\frac{1}{z(z^2-2z+2)}\right|$$

$$|z^2-2z+2| \geq R^2 - |2z-2| \qquad |z^2-2z+2| \geq R^2-2R-2$$

$$||2z-2|| \leq 2R+2$$

$$\Downarrow$$

$$M \leq \frac{1}{R(R^2-2R-2)} \leftarrow$$

ادامه ج-

$$\longrightarrow \left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{R(R^2-2R-2)}$$

$$R \longrightarrow \infty$$

$$\left|\int_{C_\infty} f(z)\,dz\right| = 0 \longrightarrow$$

$$\underbrace{\oint_C f(z)\,dz}_{\frac{\pi(1-i)}{2}} = \underbrace{\int_{C_R} f(z)\,dz}_{0} + \underbrace{\int_{C_\epsilon} f(z)\,dz}_{-\frac{\pi}{2}i} + \int_{-\infty}^{\infty} f(x)\,dx$$

$$\longrightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2}}$$